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Exact and semiclassical approach to a class of singular integral operators arising in fluid mechanics and quantum field theory

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Abstract

A class of singular integral operators, encompassing two physically relevant cases arising in perturbative QCD and in classical fluid dynamics, is presented and analysed. It is shown that three special values of the parameters allow for an exact eigenfunction expansion; these can be associated with Riemannian symmetric spaces of rank 1 with positive, negative or vanishing curvature. For all other cases an accurate semiclassical approximation is derived, based on the identification of the operators with a peculiar Schroedinger-like operator.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

It has recently been realized that a special kind of singular integral equation arising in the study of jet production ($e^+e^- \to q\bar{q}$ + anything) [1, 2] bears a striking similarity to another equation[4] introduced 40 years ago by Tuck [3] in the context of laminar flows around slender bodies. In this paper we describe a general two-parameter family of integral operators which reduce to Tuck's and Marchesini–Mueller's (hereafter MM) case for special values of the

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[4] We are indebted to R A Askey for pointing out to us the reference to Tuck's paper.

parameters:

$$(K_{\alpha\beta}\,\phi)(x) \equiv \int_{-1}^{1} \frac{\phi(x)-\phi(y)}{|x-y|}\,\mathrm{d}y + ((1-\alpha)\log(1+x) + (1-\beta)\log(1-x))\phi(x).$$

We shall show that the spectral problem can be solved exactly in three cases, connected to the three distinct symmetric spaces of rank 1 (with curvature 1, 0, -1) and they correspond to $(\alpha, \beta) = (1, 1), (0, 0), (0, 1)$, respectively. The first case [3] has discrete spectrum and it is (unitarily equivalent to) a function of the Laplacian on the sphere restricted to the axially symmetric sector. The second case has continuous spectrum and it is (unitarily equivalent to) a function of the Laplacian on the real line. The third case is equivalent to Marchesini–Mueller's operator and it is (unitarily equivalent to) a function of the radial Laplacian on the hyperbolic plane. Tuck's case corresponds to $\mathcal{T} \equiv \frac{1}{2}K_{22}$ and no exact solution is presently known. To it we can nonetheless apply a semiclassical approximation (WKB) which will be derived in general for any positive value of the parameters; the spectrum is purely discrete in this case and it is approximated by

$$\kappa_n^{(\alpha,\beta)} \approx 2\left[\log\left(\pi\left(n+\frac{1}{2}\right)\right) - \log\left(\frac{\Gamma(\alpha/2)\Gamma(\beta/2)}{\Gamma((\alpha+\beta)/2)}\right) + \left(1-\frac{1}{2}(\alpha+\beta)\right)\log 2 + \gamma_E\right] \quad (1)$$

(γ_E is Euler's constant), which in particular gives Tuck's eigenvalues to a very good accuracy (see table 1).

This paper is organized as follows. In section 2 we introduce the special problem related to jet physics and show that it is unitarily equivalent to $K_{0\,1}$. In section 3 we identify a second-order differential operator $\mathcal{L}$ commuting with $K_{0\,1}$ and determine its eigenfunction expansion. Moreover, a first-order differential operator ℓ is shown to commute with the operator $K_{0\,0}$, and also in this case we can obtain the spectral representation which is used in section 4 to introduce another representation of the operators, equivalent to Schroedinger's equation with a kinetic energy given by $g(p)$, where p is the momentum operator $-\mathrm{i}\,\mathrm{d}/\mathrm{d}u$ and g is essentially Lipatov's function. In this representation it is easy to derive qualitative properties of the operator K and to set up the semiclassical approximation. We also derive the boundary behaviour of eigenfunctions in the general case. In appendix A we show that the operator $\mathcal{L}$ commuting with $K_{0\,1}$ is indeed equivalent to the Laplace operator on the hyperbolic plane, a fact which gives us valuable information on the eigenfunction expansion (completeness, spectral measure). In appendix B we give an easy proof of Tuck's result about $K_{1\,1}$ which is essential to the developments of section 3.

2. Marchesini–Mueller's equation

Marchesini and Mueller [1] introduced an equation for the multiplicity of quark–antiquark pairs in electron–positron collisions. As a function of energy, the multiplicity density satisfies an evolution equation given by

$$\frac{\partial u(\tau,\xi)}{\partial\tau} = \int_0^1 \frac{\mathrm{d}\eta}{1-\eta}\left[\frac{u(\tau,\eta\xi)}{\eta} - u(\tau,\xi)\right] + \int_\xi^1 \frac{\mathrm{d}\eta}{1-\eta}[u(\tau,\xi/\eta) - u(\tau,\xi)]$$

where τ is the logarithm of the energy in the center of mass and $\xi = \frac{1}{2}(1-\cos\theta)$, with θ the angle between the two jets emerging from the electron–positron collision. Knowing the multiplicity at low energy, its energy dependence can be calculated at all higher energies by QCD perturbation theory and the result, in a special regime, gives the integral equation above. For details see [1]. It is formally very similar to the so-called BFKL equation [4]. The unknown $u(\tau, \xi)$ is defined for $\xi \in (0, 1)$, and it vanishes at $\xi = 0$ to ensure convergence. The initial value problem is solved if we can find the spectral decomposition of the operator on the rhs.

It is a matter of simple algebra to show that actually the equation can be recast into the form

$$\frac{\partial \phi}{\partial \tau} = -K_{0\,1}\phi$$

by performing the following transformation:

$$u(\tau, \xi) = \mathrm{e}^{-\log 2\tau} \xi \phi(\tau, 2\xi - 1).$$

The operators $K_{1\,1}$ and $K_{0\,1}$ play a central role in the following. Hence we introduce a special notation for them:

$$\mathcal{H} \equiv \tfrac{1}{2} K_{1\,1}$$
$$\mathcal{M} \equiv K_{0\,1} - \log 2 = 2\mathcal{H} + \log \tfrac{1}{2}(1 + x).$$

It is known [3] that $\mathcal{H}$ is diagonal in the basis of Legendre polynomials and its discrete eigenvalues are given by the harmonic numbers

$$\mathcal{H} P_n = \mathfrak{h}_n P_n, \qquad \mathfrak{h}_n = \begin{cases} 0 & \text{for} \quad n = 0 \\ \sum_{j=1}^{n} \frac{1}{j} & \text{for} \quad n > 0 \end{cases}$$

(a simple proof of this result can be found in appendix B). One could study the general spectral problem for (α, β) close to $(1, 1)$, e.g. by perturbation theory. However, for arbitrary values of the parameters a different approach is needed. It has been shown in [2] that the operator $\mathcal{M}$ has actually a continuous spectrum, and the eigenfunctions can be identified with hypergeometric functions. The result is obtained by an expansion starting from a combination of phase-shifted plane waves. The expansion can be pushed to all orders, and the resulting series is convergent to a hypergeometric function which can be identified with Legendre functions. The spectral decomposition of $\mathcal{M}$ is reduced to the classical *Mehler–Fock* transform. We shall come back to these facts in the appendix. Here we want to show how this exact result can be derived without any approximate procedure, by looking for a local (differential) operator commuting with $\mathcal{M}$. This will give an alternate more rigorous proof of the solution.

3. Commutativity with differential operators and exact solution

3.1. The MM operator

The easiest way to solve MM equation is to find a differential operator $\mathcal{L}$ that commutes with $\mathcal{M}$:

$$[\mathcal{L}, \mathcal{M}] = [\mathcal{L}, 2\mathcal{H}] + [\mathcal{L}, \log(1 + x)] = 0.$$

It is convenient to look for the operator $\mathcal{L}$ such that $\mathcal{L}$ as well as $[\mathcal{L}, \log(1+x)]$ acts in a simple way on $P_n(x)$; taking into account the known properties of Legendre polynomials

$$\mathcal{L}_0 P_n \equiv \left[(1 - x^2)\partial_x^2 - 2x\partial_x\right] P_n(x) = -n(n + 1) P_n(x)$$
$$x P_n = \frac{n + 1}{2n + 1} P_{n+1} + \frac{n}{2n + 1} P_{n-1}$$
$$-(1 - x^2)\partial_x P_n = \frac{n(n + 1)}{2n + 1}(P_{n+1} - P_{n-1}),$$

the problem will reduce to a purely algebraic one. Since the MM equation has a singularity at $x = -1$ we will search $\mathcal{L}$ in the form

$$\mathcal{L} = (1+x)\mathcal{L}_0 + a(1-x^2)\partial_x + b(1+x).$$

By construction, this operator acts in a simple way on $P_n(x)$. Namely,

$$\mathcal{L}P_n(x) = A_n P_{n+1} + B_n P_n + C_{n-1} P_{n-1}$$

where[5]

$$\begin{cases} (2n+1)A_n = -n(n+1)^2 - an(n+1) + b(n+1) \\ (2n+1)C_{n-1} = -n^2(n+1) + an(n+1) + bn \end{cases} \tag{2}$$

(and B_n does not enter in what follows). The action of the commutator $[\mathcal{L}, \mathcal{H}]$ on P_n is then given by

$$[\mathcal{L}, \mathcal{H}]P_n = \mathfrak{h}_n \mathcal{L}P_n - \mathcal{H}\mathcal{L}P_n = \frac{1}{n}C_{n-1}P_{n-1} - \frac{1}{n+1}A_n P_{n+1}. \tag{3}$$

The commutator $\mathcal{C} \equiv [\mathcal{L}, \log(1+x)]$ can also be easily calculated:

$$\mathcal{C} = 2(1-x^2)\partial_x - 2x + (a-1)(1-x). \tag{4}$$

Note that there are no diagonal terms coming from equation (3), hence the only diagonal contribution to the commutator comes from the last term in $\mathcal{C}$, which immediately implies $a = 1$. Using the properties of P_n, we can write

$$\mathcal{C}P_n = R_n P_{n+1} + S_{n-1} P_{n-1}$$

with

$$R_n = -2\frac{(n+1)^2}{2n+1}, \qquad S_{n-1} = 2\frac{n^2}{2n+1}.$$

Now we can check the commutativity of $\mathcal{L}$ and $\mathcal{M}$:

$$\frac{2}{n+1}A_n + R_n = 0; \qquad \frac{2}{n}C_{n-1} + S_{n-1} = 0.$$

From the first equation we have

$$A_n = -\frac{n+1}{2}R_n = -\frac{(n+1)^3}{2n+1}; \tag{5}$$

on the other hand we have from equation (2)

$$A_n = -\frac{(n+1)(n^2+2n-b)}{2n+1}$$

which fixes $b = -1$. The equation for C_{n-1} is automatically satisfied. Note that for $K_{\alpha 1}$ with $\alpha > 0$ one would find a coefficient different from the one in front of A_n in equation (5), hence no solution. $\mathcal{M}$ is therefore the only operator in the family $K_{\alpha\beta}$ which allows a commuting differential operator of the form $\mathcal{L}$. The other two cases, alluded to in the introduction, are connected to a different choice of $\mathcal{L}$ and will be discussed later.

Now we can find the eigenfunctions for the MM equation. They satisfy the differential equation

$$\mathcal{L}\phi = \lambda\phi.$$

[5] Our convention for the indices is the natural one if we think of the rhs as the action on the left by a tridiagonal matrix with vectors $[C, B, A]$ along the diagonal.

We should look for the solutions satisfying $|\phi(x)| \to |1+x|^{-1/2}$ at $x \to -1$ and which are finite at $x \to 1$. It is convenient to parametrize $\lambda = -1/2 - 2k^2$. Then the equation for ϕ can be written as

$$\left[(1-x^2)\partial_x^2 + (1-3x)\partial_x - 1 + \frac{1+4k^2}{2(1+x)}\right]\phi = 0.$$

With the substitution

$$\phi = (1+x)^\alpha \psi(x); \qquad \alpha = -1/2 + \mathrm{i}k$$

we get

$$\left[(1-x^2)\partial_x^2 + \{2\alpha + 1 - (3+2\alpha)x\}\partial_x - (\alpha+1)^2\right]\psi(x) = 0.$$

Setting $x = 1 - 2y$, we obtain

$$\left[y(1-y)\partial_y^2 + \{1 - (3+2\alpha)y\}\partial_y - (\alpha+1)^2\right]\psi(y) = 0$$

which is a hypergeometric equation with $a = b = \alpha + 1 = 1/2 + \mathrm{i}k$ and $c = 1$. The solution which is finite at $y = 0$ $(x = 1)$ and $y = 1$ $(x = -1)$ has the form

$$\psi(y) = F\left(\frac{1}{2} + \mathrm{i}k, \frac{1}{2} + \mathrm{i}k, 1, y\right);$$

$$\phi(k,x) = (1+x)^{-1/2+\mathrm{i}k} F\left(\frac{1}{2} + \mathrm{i}k, \frac{1}{2} + \mathrm{i}k, 1, \frac{1-x}{2}\right).$$

If we re-introduce the variable $\xi = (1+x)/2$ the result can be written in the form

$$\phi(k,\xi) = C\xi^{-1/2+\mathrm{i}k} F\left(\tfrac{1}{2} + \mathrm{i}k, \tfrac{1}{2} + \mathrm{i}k, 1, 1-\xi\right). \tag{6}$$

To calculate the eigenvalue κ of MM as a function of k (the dispersion relation) we may use the fact that $\mathcal{H}$ annihilates the constant and is symmetric. It follows

$$\kappa(k) \int \phi(k,\xi)\,\mathrm{d}\xi = \int \log\xi\,\phi(k,\xi)\,\mathrm{d}\xi. \tag{7}$$

This integral can be calculated and is given by a combination of *digamma* functions known as *Lipatov's function* [4]

$$\begin{aligned}\kappa(k) &= \psi\left(\tfrac{1}{2} + \mathrm{i}k\right) + \psi\left(\tfrac{1}{2} - \mathrm{i}k\right) - 2\psi(1) \\ &= -4\log 2 + 14\zeta(3)k^2 - 62\zeta(5)k^4 + O(k^6)\end{aligned} \tag{8}$$

$(\psi(z) = \mathrm{d}\log\Gamma(z)/\mathrm{d}z)$. The evolution equation for the MM equation can now be solved by expanding $u(\tau,\xi)$ on the continuous basis $\xi\phi(k,\xi)$. The spectral measure which defines the eigenfunction expansion can be taken by [5] (see also appendix).

3.2. *The case* $\alpha = \beta = 0$

We note that another operator, namely $2\mathcal{H} + \log(1-x^2) = K_{00}$, also commutes with a differential operator. In this case it is easy to prove that it commutes with a first-order differential operator[6]

$$\ell = \mathrm{i}[-(1-x^2)\partial_x + x] = -\mathrm{i}\sqrt{1-x^2}\,\partial_x\sqrt{1-x^2}. \tag{9}$$

The eigenfunctions in this case have a simple form

$$\begin{aligned}\ell\phi(k,x) &= k\phi(k,x) \\ \phi(k,x) &= (1+x)^{(\mathrm{i}k-1)/2}(1-x)^{-(\mathrm{i}k+1)/2}.\end{aligned} \tag{10}$$

[6] It will be noted that this operator is a multiple of the commutator $\mathcal{C}$ of equation (4).

The eigenvalue $g(k)$ of $K_{00} = 2\mathcal{H} + \log(1-x^2)$ belonging to these eigenfunctions can be calculated exactly in the same way as before (see equation (7)) and it has the form

$$g(k) = \psi\left(\tfrac{1}{2}(1+\mathrm{i}k)\right) + \psi\left(\tfrac{1}{2}(1-\mathrm{i}k)\right) - 2\psi(1) + 2\log 2$$

(which again can be reduced to Lipatov's function). This means that K_{00} coincides with a function of the first-order differential operator, namely

$$2\mathcal{H} + \log(1-x^2) = g(\ell). \tag{11}$$

In the following section we will use this representation to derive a semiclassical approximation and for the analysis of the asymptotic behaviour of the solutions $\phi(x)$ near the boundary points $x = \pm 1$.

4. The Schroedinger representation and semiclassical analysis

4.1. The semiclassical spectrum

We can use the representation (11) for the operator $2\mathcal{H}+\log(1-x^2)$ in terms of the first-order differential operator to transform our integral equation in the form of a Schroedinger equation (with unusual kinetic term) which is convenient for the semiclassical analysis. Consider the equation

$$K_{\alpha\beta}\phi = \{2\mathcal{H} + [(1-\alpha)\log(1+x) + (1-\beta)\log(1-x)]\}\phi = \kappa\phi. \tag{12}$$

This equation can be rewritten in terms of Schroedinger equation. Namely if we do the substitutions $x = \tanh u$ and $\phi = \cosh(u)\Psi(u)$ the last equation can be rewritten as

$$g(-\mathrm{i}\partial_u)\Psi(u) - [\alpha\log(1+\tanh u) + \beta\log(1-\tanh u)]\Psi = \kappa\Psi. \tag{13}$$

In the free case $\alpha = \beta = 0$ we have the plane waves solutions corresponding to the functions (10) after this substitution. In the case $\alpha = \beta = 1$ we have $\Psi_{1\,1} = P_n(\tanh u)/\cosh u$; this means that by identifying $\tanh u \equiv \cos\vartheta$, and modulo a similarity transformation, this case is related to the Laplace operator on the two-dimensional sphere. Finally, the case $\alpha = 0$, $\beta = 1$ corresponds to the MM equation, where also the exact solution is known.

It is convenient to slightly modify the function $g(z)$ and the eigenvalue κ by adding a constant shift. Let us introduce $G(z)$ and κ' by

$$G(z) = g(z) + 2\psi(1); \qquad \kappa' = \kappa + 2\psi(1) \tag{14}$$

where $\psi(1) = -\gamma_E$. Then equation (13) can be rewritten as

$$(G(p) + V(u))\Psi(u) = \kappa'\Psi(u) \tag{15}$$

where $p = -\mathrm{i}\partial_u$ and $V = -\alpha\log(1+\tanh u) - \beta\log(1-\tanh u)$.

We shall need the following asymptotic behaviour of $G(p)$:

$$G(p) = \begin{cases} G(0) + 7/2\zeta(3)p^2 + O(p^4), & (p \sim 0) \\ \log p^2 + O(1/p^2), & (p \to \infty). \end{cases} \tag{16}$$

The operator $G(p)$ takes on the role of the kinetic energy and is equivalent to the usual operator of non-relativistic quantum mechanics in the low energy limit. It follows that the operator $G(p) + V(u)$ has a discrete spectrum for $\beta \geqslant \alpha > 0$ (by symmetry we can restrict to the sector $\beta \geqslant \alpha$ with no loss of generality), while it has continuum spectrum for $\beta \geqslant \alpha = 0$. Let us note that in this picture Tuck's operator $\mathcal{T}$ ($\alpha = \beta = 2$) is qualitatively very similar to the 'trivial' case ($\alpha = \beta = 1$) which corresponds to operator $\mathcal{H}$.

Table 1. The WKB spectrum of $\mathcal{T}$ and $\mathcal{H}$.

n	Reference [6]	WKB	$\mathfrak{h}_n$	WKB
0	0.2332	0.3357	0	−0.116
1	1.4437	1.4343	1	0.9827
2	1.9409	1.9451	1.5	1.4935
3	2.2833	2.2816	1.8333	1.8300
4	2.5317	2.5329	2.0833	2.0813
5	2.7342	2.7335	2.2833	2.2820
6	2.9000	2.9006	2.4500	2.4490
7	3.0440	3.0437	2.5929	2.5921
8	3.1686	3.1689	2.7179	2.7173
9	3.2803	3.2801	2.8290	2.8285

The 'Schroedinger' representation can be used in different ways. In particular, we can exploit this representation to compute the semiclassical approximation to the eigenvalues and eigenfunctions. We note that near the turning points, where the kinetic term is small we can apply the standard WKB approach. This leads to the following (Bohr–Sommerfeld) semiclassical approximation to the eigenvalues:

$$\int_a^b G^{-1}(\kappa' - V)\,\mathrm{d}u = \pi(n + 1/2) \tag{17}$$

where G^{-1} is the inverse function of G and a, b are the turning points. It follows from equation (16) that the inverse function is approximately $G^{-1}(\xi) = \exp(\xi/2) + O(\exp(-\xi/2))$. In the region where the semiclassical approximation works we can neglect all asymptotic terms besides the first one. In the main approximation we can also put $a = -\infty, b = +\infty$. Then equation (17) can be rewritten as

$$\int_{-\infty}^{+\infty} \exp\left(\tfrac{1}{2}(\kappa' - V)\right)\mathrm{d}u = \pi(n + 1/2). \tag{18}$$

This integral can be easily calculated and we derive for the spectrum of the operator $K_{\alpha\beta}$:

$$\kappa_n^{(\alpha,\beta)} \approx 2\left[\log(\pi(n + 1/2)) - \log(B(\alpha/2, \beta/2)) + \left(1 - \tfrac{1}{2}(\alpha + \beta)\right)\log 2 + \gamma_E\right],$$

where B is Euler's 'beta' function, i.e. we arrive at equation (1). This equation gives the standard approximation for the harmonic numbers $\mathfrak{h}_n$ (the eigenvalues of $\frac{1}{2}K_{1\,1}$) up to $O(1/n^2)$, while for the eigenvalues of Tuck's operator $\mathcal{T} = \frac{1}{2}K_{22}$ it gives

$$\tfrac{1}{2}\kappa_n^{(2,2)} \approx \log(\pi(n + 1/2)) - \log 2 + \gamma_E. \tag{19}$$

The semiclassical formulae are contrasted with the numerical or exact eigenvalues in table 1 for operators $\mathcal{T}$ and $\mathcal{H}$ respectively.

The semiclassical eigenfunctions $\Psi^{(sc)}(u)$ can be written in the form

$$\Psi_n^{(sc)}(u) = A\sin\left(\int_{-\infty}^{u} \exp\left\{\tfrac{1}{2}(\kappa_n' - V)\right\}\mathrm{d}u + \pi/4\right)\exp(-V(u)/4)$$

where A is a normalization factor and $\kappa_n' = \kappa_n - 2\gamma_E$. The integral gives the incomplete beta function which reduces to elementary transcendentals in the special cases $\alpha = \beta = 1$ and $\alpha = \beta = 2$ corresponding to the operators $\mathcal{H}$ and $\mathcal{T}$ respectively. In the first case we have

$$\Psi_n^{(sc)}(u) = A_1 \frac{\sin[(2n + 1)\tan^{-1}(\mathrm{e}^u) + \pi/4]}{\sqrt{\cosh u}} \tag{20}$$

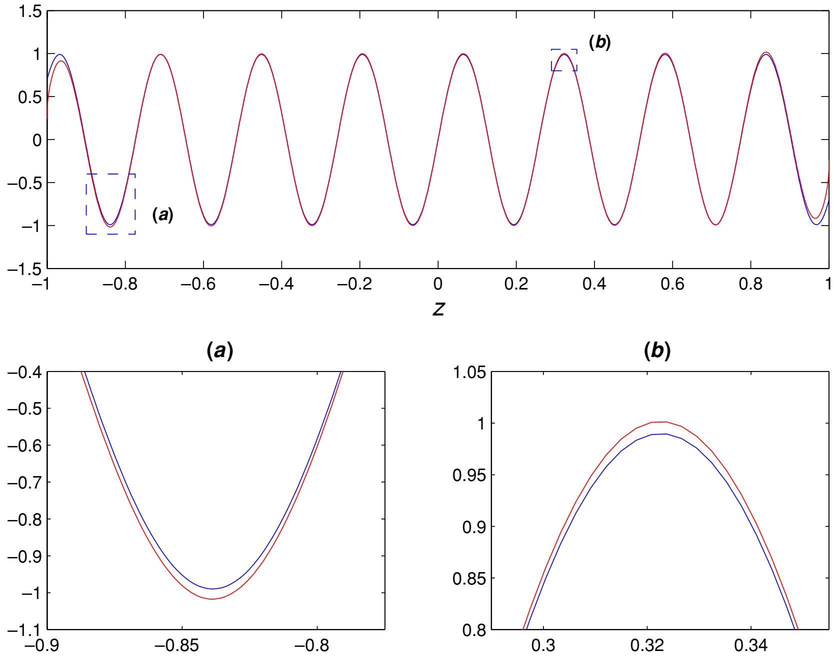

Figure 1. An example of semiclassical wavefunction (equation (22)) for the $\mathcal{T}$ operator, $n = 15$.

with $A_1 = \left(\frac{1}{2}\pi + \frac{1}{2n+1}\right)^{-1/2}$. In the case corresponding to Tuck's operator $\mathcal{T}$ we obtain

$$\Psi_n^{(sc)}(u) = A_2 \frac{\sin[\pi/2(n+1/2)(\tanh u + 1) + \pi/4]}{\cosh u} \tag{21}$$

with $A_2 = \left(1 + \frac{2/\pi}{2n+1}\right)^{-1/2}$. If we rewrite these functions in terms of original variables $x = \tanh u$ and functions $\phi(x) = \cosh(u)\Psi(u)$ then expression (20) gives the well-known large n asymptotics of Legendre polynomials:

$$\phi_n(\cos\theta) \sim \frac{\sin[(n+1/2)\theta + \pi/4]}{\sqrt{\sin\theta}}.$$

For Tuck's case ($\alpha = \beta = 2$), the semiclassical wavefunctions have a very simple form

$$\phi_n(x) = A\sin[\pi/2(n+1/2)(x+1) + \pi/4] \tag{22}$$

and they give a rather accurate description (figure 1) of the true eigenfunctions which can be easily computed numerically (i.e. by using the spectral representation for the operator $\mathcal{H}$ on the Legendre basis).

4.2. Boundary behaviour

The 'Schroedinger' representation (15) can be used to derive the asymptotic behaviour of the eigenfunctions $\phi(x)$ at the singular points $x \to \pm 1$. Namely we show that these asymptotics have the form

$$\phi_{\alpha\beta} \sim |\log(1+x)|^{d_\alpha}|\log(1-x)|^{d_\beta} \tag{23}$$

where

$$d_\alpha = \frac{1}{\alpha} - 1; \qquad d_\beta = \frac{1}{\beta} - 1. \tag{24}$$

We note that the asymptotic behaviour of the function $\phi(x)$ at $x \to \pm 1$ can be derived from the asymptotics of the function $\Psi(u)$ at $u \to \pm\infty$. Consider, for example, this asymptotics at $u \to +\infty$ $(x \to 1)$. In this limit the potential term V has the form

$$V = 2\beta u - 2\alpha \log 2 + O(\mathrm{e}^{-2u}).$$

We can neglect all terms in this expansion besides the linear one. In this way the problem is reduced to the calculation of the asymptotics at $u \to \infty$ of the solution of equation (15) with a linear potential. For this purpose it is convenient to rewrite this equation in the Fourier representation. It follows from the explicit form of function $G(p)$ that the Fourier transform $\widetilde{\Psi}(p)$ of $\Psi(u)$ satisfies the first-order differential equation:

$$-2\beta\mathrm{i}\partial_p\widetilde{\Psi}(p) + [\psi(1/2 + \mathrm{i}p/2) + \psi(1/2 - \mathrm{i}p/2) + \log 4]\widetilde{\Psi}(p) = \kappa'\widetilde{\Psi}(p).$$

The solution of this equation has the form

$$\widetilde{\Psi}(p) = \left(2^{-\mathrm{i}p}\frac{\Gamma\left(\frac{1}{2}(1-\mathrm{i}p)\right)}{\Gamma\left(\frac{1}{2}(1+\mathrm{i}p)\right)}\right)^{1/\beta} \exp(\mathrm{i}\kappa' p/2\beta). \qquad (25)$$

The asymptotics of the function

$$\Psi(u) = \int \exp(-\mathrm{i}pu)\widetilde{\Psi}(p)\,\mathrm{d}p \qquad (26)$$

is determined by the nearest singularity of the function $\widetilde{\Psi}(p)$ in the lower half plane at the point $p = -\mathrm{i}$. For non-integer $1/\beta$ this singularity is a branching point. The standard estimation of the corresponding contribution gives

$$\Psi(u) \to u^{d_\beta}\,\mathrm{e}^{-u}(1 + O(1/u)); \qquad u \to \infty.$$

Taking into account that $x = \tanh u$ and $\phi(x) = \cosh(u)\Psi(u)$ as well as the α, β symmetry of the equation we arrive at equations (23), (24). It follows from equation (25) that this asymptotics takes place in a rather narrow region $u \gg \kappa'/2\beta$ or $|\log(1-x)| \gg \kappa'/\beta$ (for $x \to 1$).

We note that for $\beta = 1$ the integral (26) can be calculated explicitly and we can derive the exact wavefunctions in the potential $V = 2u$. They have the form

$$\Psi(u) = yJ_0(2y)$$

where $y = \exp(-u + \kappa'/2)$ and $J_0(z)$ is the Bessel function.

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Appendix A. The connection to hyperbolic plane

The eigenfunctions $\phi(k, x)$ given by equation (6) are related to Legendre functions with complex index. To see this, let us revert to the original form of the MM equation for which the eigenfunctions are given by

$$u(k, x) = x^{1/2+\mathrm{i}k} F\left(\tfrac{1}{2} + \mathrm{i}k, \tfrac{1}{2} + \mathrm{i}k, 1, 1 - x\right).$$

By using well-known properties of Jacobi functions (see [5]) we find that $u(k, x)$ can be identified with the Jacobi function $\phi_{2k}^{(0,0)}(t)$ with $x = 1/\cosh^2(t)$. Hence it follows that $u(k, x) = P_{-1/2+\mathrm{i}k}(2/x - 1)$. Now, it is well known that Legendre functions of this kind appear as spherical functions on the hyperbolic plane[7], i.e. they are eigenfunctions of the radial part of the Laplace operator in the case of the two-dimensional homogeneous Riemannian space with constant negative curvature. In Gaussian coordinates, $\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r\, \mathrm{d}\varphi^2$, the radial part of the Laplacian Δ_r is given by

$$\Delta_r = (\mathrm{d}/\mathrm{d}r)^2 + \coth r\, \mathrm{d}/\mathrm{d}r.$$

and we can immediately check that $(\Delta_r + (1/4 + k^2))P_{-1/2+\mathrm{i}k}(\cosh r) = 0$. It is then natural to conclude that there must exist a map $x \to r$ and a suitable similarity transformation which connects the operator $\mathcal{L}$ of section 3 to Δ_r. From the expression of $u(k, x)$ in terms of Legendre functions, the map is given by $\cosh r = 2/x - 1 = 4/(z+1) - 1$, where z is the variable which enters the definition of $\mathcal{L}$. The similarity transformation is simply

$$\tfrac{1}{2}\mathcal{L} \equiv (1 + \cosh r)\Delta_r(1 + \cosh r)^{-1}.$$

Having established this connection, the explicit eigenfunction expansion comes for free in terms of Mehler–Fock transform

$$\begin{cases} u(x) = \int_0^\infty P_{-1/2+\mathrm{i}k}\left(\frac{2}{x} - 1\right)c(k)\,\mathrm{d}k, & (0 < x < 1) \\ c(k) = k \tanh \pi k \int_1^\infty u\left(\frac{2}{1+t}\right) P_{-1/2+\mathrm{i}k}(t)\,\mathrm{d}t & \end{cases}$$

which can be used to solve the evolution in τ for the MM equation [2].

Appendix B

We give a simple proof of an old result due to Tuck [3].

Theorem. $\frac{1}{2}K_{11}$ *has the Legendre polynomials* $P_n(x)$ *as eigenvectors with eigenvalues the* harmonic sums $\mathfrak{h}_n$.

Proof. By computing $K_{11}p_n$ with $p_n(x) \equiv x^n$ we find

$$\begin{aligned}(K_{11}p_n)(x) &= \int_{-1}^1 \mathrm{d}y \frac{x^n - y^n}{|x - y|} \\ &= \left(\int_{-1}^x - \int_x^1\right) \mathrm{d}y \sum_{k=1}^n y^{k-1}x^{n-k} \\ &= 2\mathfrak{h}_n x^n - \sum_{k=1}^n \frac{1 + (-1)^k}{k} x^{n-k},\end{aligned}$$

hence K_{11} leaves each subspace $\mathcal{P}_n$ of polynomials of degree n invariant for any n. Its matrix representation is *upper triangular* and its eigenvalues are found on the diagonal by inspection.

[7] See e.g. [7, 8].

Since K_{11} is symmetric with respect to the inner product $\langle p_1, p_2\rangle = \int_{-1}^{1} \mathrm{d}x\, p_1(x)p_2(x)$, its eigenvectors are orthogonal, and hence they are the Legendre polynomials. $\square$

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